Quantum Learning of Classical Stochastic Processes: The Completely-Positive Realization Problem

Alex Monras*[1] and Andreas Winter†[1,2]

1 Física Teòrica: Informació i Fenomens Quàntics, Universitat Autònoma de Barcelona, ES-08193 Bellaterra (Barcelona), Spain

2 ICREA – Institució Catalana de Recerca i Estudis Avançats, Pg. Lluis Companys 23, ES-08010 Barcelona, Spain

Abstract

Among several tasks in Machine Learning, is the problem of inferring the latent variables of a system and their causal relations with the observed behavior. A paradigmatic instance of such problem is the task of inferring the Hidden Markov Model underlying a given stochastic process. This is known as the positive realization problem (PRP) [3] and constitutes a central problem in machine learning. The PRP and its solutions have far-reaching consequences in many areas of systems and control theory, and is nowadays an important piece in the broad field of positive systems theory [21].

We consider the scenario where the latent variables are quantum (e. g., quantum states of a finite-dimensional system), and the system dynamics is constrained only by physical transformations on the quantum system. The observable dynamics is then described by a quantum instrument, and the task is to determine which quantum instrument – if any – yields the process at hand by iterative application.

We take as a starting point the theory of quasi-realizations, whence a description of the dynamics of the process is given in terms of linear maps on state vectors and probabilities are given by linear functionals on the state vectors. This description, despite its remarkable resemblance with the Hidden Markov Model, or the iterated quantum instrument, is however devoid from any stochastic or quantum mechanical interpretation, as said maps fail to satisfy any positivity conditions. The Completely-Positive realization problem then consists in determining whether an equivalent quantum mechanical description of the same process exists.

We generalize some key results of stochastic realization theory, and show that the problem has deep connections with operator systems theory, giving possible insight to the lifting problem in quotient operator systems. Our results have potential applications in quantum machine learning, device-independent characterization and reverse-engineering of stochastic processes and quantum processors, and more generally, of dynamical processes with quantum memory [16, 17].

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* This research was partly undertaken while A. M. was at the Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, Singapore 117542, Singapore.

† This research was partly undertaken while A. W. was at the Department of Mathematics, University of Bristol, Bristol BS8 1TW, United Kingdom.

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1 Introduction

Let $\mathcal{M}$ be an alphabet with $|\mathcal{M}| = m$ symbols and let $\mathcal{M}^\ell$ be the set of words of length ℓ. Let $\mathcal{M}^*$ be the free monoid generated by $\mathcal{M}$

$$\mathcal{M}^* = \bigcup_{\ell \geq 0} \mathcal{M}^\ell. \tag{1}$$

We will be concerned with stochastic processes defined on sequences of random variables over $\mathcal{M}$, i. e., stationary probability measures over $\mathcal{M}^*$. We assume throughout that p is a stationary stochastic process on $\mathcal{M}^\infty$, namely,

$$p(\mathbf{u}) \equiv p(\mathcal{Y}_t = u_1, \mathcal{Y}_{t+1} = u_2, \ldots, \mathcal{Y}_{t+\ell-1} = u_\ell), \quad \mathbf{u} = (u_1, \ldots, u_\ell) \in \mathcal{M}^\ell \tag{2}$$

is independent of t. We will use ℓ to denote a generic length of a word $\mathbf{u}$, so that $\mathbf{u}$ can be written as $\mathbf{u} = (u_1, \ldots, u_\ell)$ instead of the more cumbersome $\mathbf{u} = (u_1, \ldots, u_{|\mathbf{u}|})$. Let p be a stationary stochastic process defined on the alphabet $\mathcal{M}$.

▶ **Definition 1.** A quasi-realization of a stochastic process is a quadruple $(\mathcal{V}, \pi, D, \tau)$ where $\mathcal{V}$ is a vector space, $\tau \in \mathcal{V}$, $\pi \in \mathcal{V}^*$ and $D : \mathcal{M}^* \to \mathcal{L}(\mathcal{V})$ is a representation of $\mathcal{M}^*$ over $\mathcal{V}$,

$$D^{(\mathbf{u})} D^{(\mathbf{v})} = D^{(\mathbf{uv})}, \quad \forall \mathbf{u}, \mathbf{v} \in \mathcal{M}^*. \tag{3}$$

In addition, the following relations hold,

$$\pi^\top \left[\sum_{u \in \mathcal{M}} D^{(u)} \right] = \pi^\top, \qquad \left[\sum_{u \in \mathcal{M}} D^{(u)} \right] \tau = \tau \tag{4}$$

and

$$p(\mathbf{u}) = \pi^\top D^{(\mathbf{u})} \tau \quad \forall \mathbf{u} \in \mathcal{M}^*. \tag{5}$$

The smallest dimensional quasi-realization admitted by p is called *regular* realization, and its dimension is the *order* of p. The regular realization is efficiently computable given the probabilities of words of length $2r - 1$, where r is the order of p [10, 24].

2 The classical learning problem

A central task in machine learning is to obtain the latent variables that account for the apparent complexity of a given process p. These variables, although not directly accessible to the observable dynamics summarize past behavior while still providing complete information about future probabilities of events. To accomplish this, one aims to find a random variable X such that the future is independent of the past, given X,

$$p(\mathbf{v}|\mathbf{u}) = \sum_X P(\mathbf{v}|X) P(X|\mathbf{u}). \tag{6}$$

However, such a decomposition to exist at any given time we require that state transition probabilities are only dependent on the generated output, $P(X_t, u_t | X_{t-1})$ in a time-invariant manner. This implies that X is markovian, and we say that p is a Hidden Markov Process. In such case, $\{X_t\}$ represents the latent variables of p, and an important problem in machine learning consists in recovering the probabilities $P(X_t, u_t | X_{t-1})$.

A process' quasi-realization constitutes an abstract model of the behavior of p. However this does not suffice to identify its latent variables, as the vector $\pi D^{(\mathbf{u})}$ does not necessarily satisfy any positivity criterion, and the maps need not be related to any stochastic transition probabilities. Moreover, the vectors $\pi D^{(\mathbf{u})}$ will potentially acquire an unbounded number of distinct values over $\mathcal{V}$, giving little insight on the essential mechanisms driving p.

A *positive* realization of p is a quasi-realization $(\mathcal{V}, \pi, D, \tau)$, such that $D^{(\mathbf{u})}$ are substochastic matrices (nonnegative matrices such that $\sum_{u\in\mathcal{M}} D^{(u)}$ is stochastic), π is the stationary distribution, and $\pi = (1, 1, \ldots, 1)$. **The Positive Realization Problem** (PRP) is the problem of finding a positive realization of a process p, given its regular realization [24].

3 The quantum learning problem

We address the natural quantum generalization of this problem, namely, when the relevant information about the past can be synthesized by a quantum state, rather than a classical random variable. This requirement, less impositive than the classical one [22], has been considered from the perspective of ϵ-machines [15], where it was shown that the statistical complexity of the system could be reduced by a quantum model. Instead, our approach focuses on the dimension of the quantum system, which can be drastically reduced once one allows for quantum states. A highly relevant example in a not too distant scenario can be found in [25].

In the quantum mechanical setting, the factorization condition Eq. (6) is replaced by

$$p(\mathbf{v}|\mathbf{u}) = \rho_{\mathbf{u}}[M^{(\mathbf{v})}], \tag{7}$$

where $\rho_{\mathbf{u}}$ represents a quantum state, and $M^{(\mathbf{v})}$ the POVM element associated with outcome $\mathbf{v}$. Future probabilities are obtained by the Born rule applied to state $\rho_{\mathbf{u}}$. The minimum dimension by which this description can be achieved is given by the positive semidefinite rank [13]. However, in addition, in order to have a physically meaningful description of the mechanisms at work, one expects that the state transition probabilities are given by physical transformations,

$$\rho_{\mathbf{u}v} = \rho_{\mathbf{u}} \circ \mathcal{E}^{(v)}, \tag{8}$$

where $\mathcal{E}^{(v)}$ are completely-positive maps, and $\sum_{v\in\mathcal{M}} \mathcal{E}^{(v)}$ is unital. The set $\{\mathcal{E}^{(v)}\}$ is called a *quantum instrument*. This problem has received little attention in the literature. It arises naturally – albeit in slight disguise – in [5], and more generally in systems identification [6, 17, 2, 7].

The completely positive realization problem (CPRP): *Given a quasi-realization of process $p(\boldsymbol{u})$, determine whether there exist a quantum instrument $\{\mathcal{E}^{(u)}\}$, and positive semidefinite ρ such that*

$$p(\boldsymbol{u}) = \rho[\mathcal{E}^{(u_1)} \circ \cdots \circ \mathcal{E}^{(u_\ell)}(\mathcal{I})], \tag{9}$$

such that $\mathcal{E} = \sum_u \mathcal{E}^{(u)}$ is completely positive and unital, and $\rho \circ \mathcal{E} = \rho$. Stochastic processes admitting a completely-positive realization are called quantum Markov chains (QMC) [1, 11].

In order to obtain necessary and sufficient conditions for p to be a QMC, we first generalize a classical result by Ito, Amari and Kobayashi [18]. The latter is the stochastic equivalent to a classic result on linear systems theory [19], *i. e.*, minimal realizations are always related by similarity transformations, and are quotients of higher-dimensional ones.

Define the $\mathcal{W} = \operatorname{span}\{\mathcal{E}^{(\mathbf{u})}(\mathcal{I})\}_{\mathbf{u}\in\mathcal{M}^*}$ as the *accessible* subspace. It is trivially stable under the action of $\mathcal{E}^{(\mathbf{u})},\ \forall \mathbf{u} \in \mathcal{M}^*$,

$$\mathcal{E}^{(\mathbf{u})}(\mathcal{W}) \subseteq \mathcal{W}. \tag{10}$$

Analogously, we consider the span of states $\widetilde{\mathcal{W}} = \operatorname{span}\{\rho \circ \mathcal{E}^{(\mathbf{u})}\}_{\mathbf{u}\in\mathcal{M}^*}$. Its annihilator, $\widetilde{\mathcal{W}}^{\perp} = \bigcap_{\sigma\in\widetilde{\mathcal{W}}} \ker\sigma$, is the *null* space, *i. e.*, the subspace which has no effect whatsoever for computing word probabilities. Also $\widetilde{\mathcal{W}}^{\perp}$ is stable under $\mathcal{E}^{(\mathbf{u})},\ \forall \mathbf{u} \in \mathcal{M}^*$,

$$\mathcal{E}^{(\mathbf{u})}(\widetilde{\mathcal{W}}^{\perp}) \subseteq \widetilde{\mathcal{W}}^{\perp}. \tag{11}$$

Define the *quotient* space $\mathcal{V}$ as the accessible space modulo its null component $K = \mathcal{W} \cap \widetilde{\mathcal{W}}^{\perp}$,

$$\mathcal{V} \equiv \frac{\mathcal{W}}{K}. \tag{12}$$

The elements of $\mathcal{V}$ are of the form $a + K,\ a \in \mathcal{W}$. Let $L : \mathcal{W} \to \mathcal{V}$ be the canonical projection onto $\mathcal{V}$,

$$\begin{array}{rccc} L: & \mathcal{W} & \to & \mathcal{V} \\ & v & \mapsto & v + K. \end{array} \tag{13}$$

Since $\mathcal{E}^{(\mathbf{u})}(K) \subseteq (K)$ let $\mathcal{D}$ be the induced quotient map $\mathcal{D}^{(\mathbf{u})} : \mathcal{V} \to \mathcal{V}$, as defined by $\mathcal{D}^{(\mathbf{u})} \circ L = L \circ \mathcal{E}^{(\mathbf{u})}$. Also, define $\tau = L(\mathcal{I})$ and π as the induced quotient functional $\pi \circ L = \rho$. Using the fact that $\rho[\ker L] = 0$ we factor through the entire set of maps $\mathcal{E}^{(u)}$,

$$\begin{aligned} p(\mathbf{u}) &= \rho \circ \mathcal{E}^{(\mathbf{u})}(\mathcal{I}) && (14a) \\ &= \pi \circ L \circ \mathcal{E}^{(\mathbf{u})}(\mathcal{I}) && (14b) \\ &= \pi \circ \mathcal{D}^{(\mathbf{u})}(\tau). && (14c) \end{aligned}$$

This, together with easily shown eigenvector relations (4) illustrate that $(\mathcal{V}, \pi, \mathcal{D}^{(u)}, \tau)$ constitute a perfectly valid quasi-realization. We call such quasi-realization the *quotient* realization. An important step is to realize that the quotient spaces of equivalent quasi-realizations are minimal and hence isomorphic.

▶ **Theorem 2.** *[18] Two quasi-realizations $\mathcal{R}_1 = (\mathcal{V}_1, \pi_1, D_1^{(u)}, \tau_1)$ and $\mathcal{R}_2 = (\mathcal{V}_2, \pi_2, D_2^{(u)}, \tau_2)$ of the same stochastic process p, not necessarily of the same dimension, have isomorphic quotient realizations $\overline{\mathcal{R}}_i = (\overline{\mathcal{V}}_i, \overline{\pi}_i, \overline{D}_i^{(u)}, \overline{\tau}_i)_{i=1,2}$, $\bar{\mathcal{V}}_1 \overset{T}{\cong} \bar{\mathcal{V}}_2$,*

$$\overline{\pi}_1^{\top} = \overline{\pi}_2^{\top} T, \tag{15}$$

$$\overline{\mathcal{D}}_1^{(u)} = T^{-1} \overline{\mathcal{D}}_2^{(u)} T, \tag{16}$$

$$\overline{\tau}_1 = T^{-1} \overline{\tau}_2. \tag{17}$$

This result follows from [18], which proves it only for the Hidden Markov Model case. The proof, however, only relies on the nonnegativity of the process' probabilities, and applies to any pair of equivalent and well-defined (in the sense that they yield the same nonnegative measure on $\mathcal{M}^*$) quasi-realizations.

This result is important in that it establishes the uniqueness of the quotient space $\mathcal{V}$, up to basis changes. Let d be the dimension of $\mathcal{W}$. As can be seen from the definition $d = \dim \mathcal{V} \leq n$, where n is the original realization's dimension. By considering the quotient of a regular realization of dimension r we get $d \leq r$. On the other hand r is a lower bound to the dimension of any quasi-realization. Thus we conclude that $d = r$, hence quotient realizations are indeed regular, and all regular realizations can be regarded as quotient realizations.

4 Semidefinite representable cones and quotient operator systems

The CPRP aims at providing a completely-positive lifting of a regular realization $\mathcal{R} = (\pi, D^{(u)}, \tau)$. As it will be shown, a necessary and sufficient condition is the existence of certain stable cones of a particular kind, containing the vector τ, and whose dual contains π. We focus on finite-dimensional liftings from an r-dimensional regular realization $\mathcal{R}$ acting on $\mathcal{V} \cong \mathbb{R}^r$ to a completely positive realization acting on $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a finite-dimensional Hilbert space, $\mathcal{H} = \mathbb{C}^n$. We use S^+ to denote the positive semidefinite cone in $\mathcal{B}(\mathcal{H})$. All cones we deal with are convex. A cone $\mathcal{C}$ is pointed iff $x \in \mathcal{C}$ and $-x \in \mathcal{C}$ implies $x = 0$ and $\mathcal{C}$ is generating if $\operatorname{span}\mathcal{C} = \mathcal{V}$. We will use calligraphic letters for subspaces of $\mathcal{B}(\mathcal{H})$, and for any given subspace $\mathcal{W}$, $\mathcal{W}^+$ will denote its intersection with S^+, $\mathcal{W}^+ = \mathcal{W} \cap \mathrm{S}^+$.

▶ **Definition 3.** Let $\mathcal{V}$ be a finite dimensional real vector space. A *semidefinite representable cone* (SDR) is a set $\mathcal{C} \in \mathcal{V}$ such that

$$\mathcal{C} = L(\mathcal{W}^+) \tag{18}$$

where $\mathcal{W} \subseteq \mathcal{B}(\mathcal{H})$ is a subspace and $L : \mathcal{W} \to \mathcal{V}$ is a linear map.

It is easy to see that pointed and generating SDR cones can always be described by subspaces $\mathcal{W}$ such that $\mathcal{W} = \operatorname{span}(\mathcal{W}^+)$ and L is a quotient map from $\mathcal{W}$ to $\mathcal{W}/K \cong \mathcal{V}$, with $K \cap \mathrm{S}^+ = \{0\}$. SDR cones are homogeneous versions of semidefinite representable sets, the feasibility regions of semidefinite programs [4].

▶ **Lemma 4.** *Let $\mathcal{I} \in \mathcal{W} \subseteq \mathcal{B}(\mathcal{H})$ and $\widetilde{\mathcal{W}} \subseteq \mathcal{B}(\mathcal{H})^*$, such that $\mathcal{W} = \operatorname{span}(\mathcal{W}^+)$ and $K = \mathcal{W} \cap \widetilde{\mathcal{W}}^\perp$ satisfies $K \cap \mathrm{S}^+ = \{0\}$. Let L be the canonical projection $L : \mathcal{W} \to \mathcal{W}/K$. Then $\mathcal{C} = L(\mathcal{W}^+)$ is a pointed, generating SDR cone, and its dual is given by*

$$\mathcal{C}^* = \widetilde{L}\big((\widetilde{\mathcal{W}} + \mathcal{W}^\perp)^+\big) \tag{19}$$

where $\widetilde{L}$ is the canonical projection $\widetilde{L} : \widetilde{\mathcal{W}} + \mathcal{W}^\perp \to (\widetilde{\mathcal{W}} + \mathcal{W}^\perp)/\mathcal{W}^\perp \cong \mathcal{V}^*$.

Since $\mathcal{I} \in \mathcal{W} \subseteq \mathcal{B}(\mathcal{H})$, $\mathcal{W}$ can be regarded as an operator system [23]. Let $\mathcal{W}_n = \mathcal{W} \otimes \mathcal{B}(\mathbb{C}^n)$ and $\mathcal{W}_n^+$ its positive cone. Likewise, given a linear map $L : \mathcal{W} \to \mathcal{V}$, let $L_n \equiv L \otimes \mathcal{I}_n : \mathcal{W}_n \to \mathcal{V}_n$. We define cones $\mathcal{C}_n$ as

$$\mathcal{C}_n = L_n(\mathcal{W}_n^+) \subset \mathcal{V}_n. \tag{20}$$

Since $K \cap \mathrm{S}^+ = \{0\}$ then $(\mathcal{V}, \mathcal{C}_n, L(\mathcal{I}))$ define a quotient operator system [20].

5 Regular quasi-realizations as quotient realizations

From Theorem 2 it follows that given a regular quasi-realization $\mathcal{R} = (\mathcal{V}, \pi, \mathcal{D}^{(u)}, \tau)$, for an equivalent completely-positive realization $\mathcal{Q} = (\mathcal{B}(\mathcal{H}), \rho, \mathcal{E}^{(u)}, \mathcal{I})$ to exist, the former must be a quotient realization of the latter. This implies several constraints on the structure of the stable subspaces of $\mathcal{Q}$, and provide necessary conditions for the feasibility of the CPRP.

For a hypothetical completely-positive realization for p, the accessible subspace $\mathcal{W} = \operatorname{span}\{\mathcal{E}^{(u)}(\mathcal{I})\}$ is an operator system in $\mathcal{B}(\mathcal{H})$, and complete positivity of $\mathcal{E}$ in $\mathcal{W}$ suffices, by virtue of Arveson's theorem, to ensure complete positivity in $\mathcal{B}(\mathcal{H})$,

$$\mathcal{E}_n(\mathcal{W}_n^+) \subseteq \mathcal{W}_n^+. \tag{21}$$

The null space $\widetilde{\mathcal{W}}^{\perp}$ of $\mathcal{Q}$, and more precisely its restriction to $\mathcal{W}$, $K = \mathcal{W} \cap \widetilde{\mathcal{W}}^{\perp}$ must also be stable under the action of $\mathcal{E}^{(\mathbf{u})}$. The quotient space is $\mathcal{V} = \mathcal{W}/K$ and the canonical projection $L : \mathcal{W} \to \mathcal{V}$ brings $\mathcal{Q}$ to $\mathcal{R}$. In particular, we have the following relations

$$\tau = L(\mathcal{I}), \tag{22a}$$

$$\pi \circ L = \rho \tag{22b}$$

which relate $\mathcal{R}$ to $\mathcal{Q}$. Under the quotient construction, the induced maps satisfy the relation

$$\mathcal{D} \circ L = L \circ \mathcal{E}. \tag{23}$$

Using the definitions of the previous section, we have

$$\mathcal{D}_n(\mathcal{C}_n) \subseteq \mathcal{C}_n, \quad \forall n \geq 1. \tag{24}$$

This is precisely the condition of complete positivity in the quotient operator system $(\mathcal{V}, \mathcal{C}_n, L(\mathcal{I}))$. Hence a necessary condition for the existence of a CP realization is that the regular realization is completely-positive with respect to a quotient operator system, together with relations

$$\tau \in \mathcal{C} \tag{25}$$

$$\pi \in \mathcal{C}^*, \tag{26}$$

which follow from (22). However, as it turns out, this condition does not suffice to guarantee existence of a completely positive lift in $\mathcal{W}$. In fact, there exist completely-positive maps in $\mathcal{V}$ which are not induced quotients of completely-positive maps in $\mathcal{W}$. To overcome this difficulty, we will not impose complete positivity in the standard operator systems sense, but instead impose a stronger condition that guarantees complete positivity in the quotient operator system $\mathcal{V}$ as well as in $\mathcal{W}$.

Let us denote $\mathcal{E}$ for an arbitrary element $\mathcal{E}^{(u)}$ and regard it as an element in $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})^*$. Maps satisfying $\mathcal{E}(\mathcal{W}) \subseteq \mathcal{W}$ and $\mathcal{E}(K) \subseteq K$ are in the subspace $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})^*$,

$$\mathcal{S} = \mathcal{W} \otimes \widetilde{\mathcal{W}} + K \otimes \mathcal{B}(\mathcal{H})^* + \mathcal{B}(\mathcal{H}) \otimes \mathcal{W}^{\perp}. \tag{27}$$

Let $\varphi : K^{\perp} \to (\mathcal{W}/K)^* = \mathcal{V}^*$ be the natural isomorphism between these two spaces, and let $\phi : \mathcal{B}(\mathcal{H})^* \to \mathcal{B}(\mathcal{H})^*/\mathcal{W}^{\perp}$ be the canonical quotient map modulo $\mathcal{W}^{\perp}$. Then define

$$\widetilde{L} : \mathcal{W}^{\perp} + \widetilde{\mathcal{W}} \xrightarrow{\phi} K^{\perp} \xrightarrow{\varphi} \mathcal{V}^*. \tag{28}$$

Now, consider the map $L \otimes \widetilde{L}$. In principle, the range of this map is not well defined in the entire $\mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H})^*$, and arbitrary extensions would be required. However, for each of these spaces is well-defined,

$$L \otimes \widetilde{L} : \quad K \otimes \mathcal{B}(\mathcal{H})^* \longrightarrow 0 \quad (K = \ker L) \tag{29}$$

$$\mathcal{B}(\mathcal{H}) \otimes \mathcal{W}^{\perp} \longrightarrow 0 \quad (\mathcal{W}^{\perp} = \ker \widetilde{L}) \tag{30}$$

$$\mathcal{W} \otimes \widetilde{\mathcal{W}} \longrightarrow \mathcal{V} \otimes \mathcal{V}^*. \tag{31}$$

We thus have that $\mathcal{D} = L \otimes \widetilde{L}(\mathcal{E}) \in \mathcal{V} \otimes \mathcal{V}^*$ is the induced quotient map. Also, completely-positive maps with these stable subspaces form a cone $\mathcal{S}^{\mathrm{CP}}$, where CP denotes intersection with the completely positive cone. Finally, we conclude that

$$\mathcal{D} \in \mathcal{P} = L \otimes \widetilde{L}\left(\mathcal{S}^{\mathrm{CP}}\right). \tag{32}$$

By the Choi-Jamiolkowski isomorphism, $\mathcal{S}^{\mathrm{CP}}$ is isomorphic the positive semidefinite subcone of some subspace of $\mathcal{B}(\mathcal{H}\otimes\mathcal{H})$, hence, $\mathcal{P}$ is semidefinite representable. One can check that $\mathcal{D}\in\mathcal{P}$ implies complete positivity in the operator system $(\mathcal{V},\mathcal{C}_n,L(\mathcal{I}))$.

Notice that the identity map is in $\mathcal{P}$ since it just corresponds to the induced map of the identity in $\mathcal{S}^{\mathrm{CP}}$, which is completely positive and satisfies all the stability conditions. In addition, other useful properties hold for $\mathcal{P}$. In particular,

- $\mathcal{P}$ is pointed.
- $\mathcal{P}$ is closed under composition, *i. e.* it is a semigroup.
- $\mathcal{C}\otimes_{\max}\mathcal{C}^*\subseteq\mathcal{P}$, where $\otimes_{\max}$ denotes the maximal tensor product, *i. e.* the convex hull of pairs of elements $\rho\otimes\sigma$, $\rho\in\mathcal{C},\sigma\in\mathcal{C}^*$.

Notice also, that given $\mathcal{P}$ and π,τ, one can obtain $\mathcal{C}$ from $\mathcal{P}$, $\mathcal{C}=\mathcal{P}\tau$.

In conclusion, the necessary conditions for the CPRP can be stated as

- $\tau\in\mathcal{C}$,
- $\pi\in\mathcal{C}^*$,
- $\mathcal{D}\in\mathcal{P}$.

with $\mathcal{P}$ of the type (32). The next section shows that these conditions are also sufficient.

6 Sufficiency of the conditions

So far we have derived a set of necessary conditions which follow from the hypothesis that an underlying completely-positive realization exists. In this section we show that these are also sufficient.

▶ **Theorem 5** (Removing spurious eigenvectors)**.** *Let $\{\mathcal{E}^{(u)}\}$ be a set of completely positive maps on $\mathcal{B}(\mathcal{H})$ with $\mathcal{E}=\sum_u\mathcal{E}^{(u)}$, and let ρ, $\mathcal{I}$ be positive semidefinite operators in $\mathcal{B}(\mathcal{H})$ such that $\mathrm{tr}[\rho\mathcal{I}]=1$. If ω is a positive semidefinite eigenvector of $\mathcal{E}$ such that $\mathrm{tr}[\rho\omega]=0$, then there is always another set of CP maps $\{\hat{\mathcal{E}}^{(u)}\}$ on $\mathcal{B}(\ker(\omega))$ and positive semidefinite operators $\hat\rho$, $\hat{\mathcal{I}}\in\mathcal{B}(\ker(\omega))$ such that*

$$\mathrm{tr}[\rho\,\mathcal{E}^{(\boldsymbol{u})}(\mathcal{I})]=\mathrm{tr}[\hat\rho\,\hat{\mathcal{E}}^{(\boldsymbol{u})}(\hat{\mathcal{I}})]\qquad\forall\boldsymbol{u}\in\mathcal{M}^*. \tag{33}$$

Proof. Let $\mathcal{P}=\ker(\omega)$ and $\mathcal{Q}=\mathrm{range}(\omega)=\mathcal{P}^\perp$ its orthogonal complement. Let P (resp. Q) be the corresponding orthogonal projection in $\mathcal{H}$, and $\Pi_\mathcal{P}=P\cdot P$, (resp. $\Pi_\mathcal{Q}$) the hereditary projection on $\mathcal{B}(\mathcal{H})$. Since ω is a positive semidefinite eigenvector, we have that $\mathcal{E}\circ\Pi_\mathcal{Q}=\Pi_\mathcal{Q}\circ\mathcal{E}\circ\Pi_\mathcal{Q}$. From positivity, this extends to all $\mathcal{E}^{(\mathbf{u})}$ and thus

$$\Pi_\mathcal{P}\circ\mathcal{E}^{(\mathbf{u})}=\Pi_\mathcal{P}\circ\mathcal{E}^{(\mathbf{u})}\circ\Pi_\mathcal{P},\quad\forall\mathbf{u}\in\mathcal{M}^*. \tag{34}$$

From orthogonality of $\rho\geq0$ and $\omega\geq0$ it follows that $\rho=\Pi_\mathcal{P}(\rho)$ and we can write

$$\begin{aligned}p(\mathbf{u})&=\mathrm{tr}[\rho\,\Pi_\mathcal{P}\mathcal{E}^{(\mathbf{u})}(\mathcal{I})]\\&=\mathrm{tr}[\rho\,\Pi_\mathcal{P}\mathcal{E}^{(u_1)}\Pi_\mathcal{P}\circ\Pi_\mathcal{P}\mathcal{E}^{(u_2)}\Pi_\mathcal{P}\cdots\Pi_\mathcal{P}\mathcal{E}^{(u_\ell)}\Pi_\mathcal{P}(\mathcal{I})].\end{aligned} \tag{35}$$

Replace $\mathcal{H}\leftarrow\mathcal{P}$, $\mathcal{B}(\mathcal{H})\leftarrow\mathcal{B}(\mathcal{P})$ and

$$\mathcal{E}^{(u)}\leftarrow\Pi_\mathcal{P}\mathcal{E}^{(u)}\Pi_\mathcal{P} \tag{36a}$$
$$\mathcal{I}\leftarrow\Pi_\mathcal{P}(\mathcal{I}) \tag{36b}$$
$$\rho\leftarrow\Pi_\mathcal{P}(\rho). \tag{36c}$$

The resulting maps are still completely positive and ρ, $\mathcal{I}$ are positive semidefinite with support in $\mathcal{B}(\mathcal{P})$, thus the new $\mathcal{I}$ has $\mathrm{tr}[\mathcal{I}\omega]=0$. In addition, from Eq. (35), they generate the same process. ◀

▶ **Theorem 6.** *Given a pseudo-realization $\mathcal{R} = (\mathcal{V}, \pi, \mathcal{D}, \tau)$, an equivalent, finite-dimensional, unital, completely-positive realization exists $(\mathcal{B}(\mathcal{H}), \rho, \mathcal{E}, \mathcal{I})$ if and only if there is an SDR cone $\mathcal{P} \subset \mathcal{V} \otimes \mathcal{V}^*$ such that*
1. $\mathcal{D}^{(u)} \in \mathcal{P},\ \forall u \in \mathcal{M}$,
2. $\tau \in \mathcal{C}$,
3. $\pi \in \mathcal{C}^*$.

where $\mathcal{C}$, $\mathcal{C}^*$ and $\mathcal{P}$ are of type (18)*,* (19) *and* (32)*, respectively.*

Proof. That the conditions are necessary was proven in the previous section. It follows from condition 1 that CP maps $\mathcal{E}^{(u)} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ can be defined such that $\mathcal{E}^{(u)}(K) \subseteq K$ and $\mathcal{E}^{(u)}(\mathcal{W}) \subseteq \mathcal{W}$, and that

$$L \circ \mathcal{E}^{(u)} = \mathcal{D}^{(u)} \circ L, \quad \forall u \in \mathcal{M}. \tag{37}$$

To lift the vectors τ and π, notice that since $\tau \in \mathcal{C}$ and $\pi \in \mathcal{C}^*$, there is $\mathcal{I} \in \mathcal{W}^+$ and $\rho \in (\mathcal{W}^\perp + \widetilde{\mathcal{W}})^+$ such that

$$\tau = L(\mathcal{I}) \tag{38}$$

$$\rho = \pi \circ L. \tag{39}$$

At this point it is easy to check that $\mathcal{D}^{(\mathbf{u})}(\tau) = \mathcal{D}^{(\mathbf{u})} L(\mathcal{I}) = L\mathcal{E}^{(u)}(\mathcal{I})$, so that

$$\pi \cdot \mathcal{D}^{(\mathbf{u})}(\tau) = \rho \circ \mathcal{E}^{(\mathbf{u})}(\mathcal{I}), \quad \forall \mathbf{u} \in \mathcal{M}^*, \tag{40}$$

However, the operators ρ and $\mathcal{I}$ are not left- and right-eigenvectors of $\mathcal{E} = \sum_{u \in \mathcal{M}} \mathcal{E}^{(u)}$, so they $(\mathcal{B}(\mathcal{H}), \mathcal{I}, \mathcal{E}, \rho)$ does not form a realization. In order to find a proper completely-positive realization, we will iteratively replace them by suitable projections by making use of Theorem 5, until the desired properties are obtained. In the process, we remove all spurious contributions to ρ and $\mathcal{I}$ until only relevant contributions to Eq. (40) remain.

STEP 1: Consider the Cesàro mean $\omega_n = \frac{1}{n} \sum_{k=1}^n \mathcal{E}^k(\mathcal{I})$. Clearly, $\omega_n \geq 0\ \forall n$. Define the ratio $\lambda = \lim_{n\to\infty} \frac{\|\omega_{n+1}\|}{\|\omega_n\|}$ so that the limit is well-defined,

$$\omega = \lim_{n\to\infty} \frac{\omega_n}{\lambda^n}. \tag{41}$$

Clearly, $\omega \geq 0$, and

$$\mathcal{E}(\omega) = \lim_{n\to\infty} \frac{1}{n\lambda^n} \sum_{k=1}^n \mathcal{E}^{k+1}(\mathcal{I}) = \lambda\omega. \tag{42}$$

At this point, two different scenarios may occur. Either $\lambda = 1$ or $\lambda > 1$. Consider first the case when $\lambda > 1$. This means that there is a contribution to $\mathcal{I}$ which grows under the action of $\mathcal{E}$, and ω captures its asymptotic behavior. One can see that

$$\begin{aligned} \mathrm{tr}[\rho\omega] &= \lim_{n\to\infty} \frac{1}{n\lambda^n} \sum_{k=1}^n \mathrm{tr}[\rho\mathcal{E}^k(\mathcal{I})] \\ &= \lim_{n\to\infty} \frac{1}{\lambda^n} \\ &= 0. \end{aligned} \tag{43}$$

Hence, by making use of Theorem 5, we can obtain a new set of CP maps $\{\mathcal{E}^{(u)}\}$, ρ and $\mathcal{I}$ such that $\mathrm{tr}[\mathcal{I}\omega] = 0$. However, ρ and $\mathcal{I}$ are still not eigenvectors. Repeat STEP 1 until $\lambda = 1$.

If $\lambda = 1$ then $\omega = \lim_{n\to\infty} \omega_n$ is well defined. Replace $\mathcal{I} \leftarrow \omega$ and proceed to STEP 2.

At each iteration of STEP 1 a new ω is obtained, orthogonal to all previous ones, and the associated eigenvalue can only be equal or decrease. The aim of this iteration is to capture the eigenspace of $\mathcal{E}$ with the largest eigenvalue and remove it without altering the resulting stochastic process $p(\mathbf{u})$.

Because $\mathcal{E}$ has only finitely many eigenvalues, eventually λ will equal 1. In that case, the resulting ω is strictly positive. Proceed to PART 2.

STEP 2: At this point $\mathcal{I}$ is an eigenvector but ρ is not. Rerurn STEP 1 with the dual realization, *i. e.*, with $(\mathcal{B}(\mathcal{H})^*, \mathcal{I}, \mathcal{E}^*, \rho)$, interchanging the roles of ρ and $\mathcal{I}$.

After STEP 2, ρ is an eigenvelue of $\mathcal{E}$ but $\mathcal{I}$ may not be. A further iteration of steps 1 and 2 will lead to further dimension reductions. Since the dimension is finite, eventually no further truncations will be necessary and both $\mathcal{I}$ and ρ will be proper left- and right-eigenvalues of $\mathcal{E}$.

Once one has iterated through STEPS 1 and 2, one has a completely-positive realization $(\rho, \mathcal{E}^{(u)}, \mathcal{I})$ with the required stability properties for ρ and $\mathcal{I}$. It just remains to ensure that $\mathcal{I} > 0$. The procedure is very similar to the one just exposed.

STEP 3: Let $\mathcal{Q} = \ker(\mathcal{I})$ and $\mathcal{P} = \mathcal{Q}^\perp = \mathrm{range}(\mathcal{I})$ its orthogonal complement. Since $\mathcal{I} \geq 0$ is an eigenvector of $\mathcal{E}$, we have that $\mathcal{E}^{(\mathbf{u})}(\mathcal{I}) \in \mathcal{B}(\mathcal{P})$, $\forall \mathbf{u} \in \mathcal{M}^*$. Hence we can make the substitutions $\mathcal{H} \leftarrow \mathcal{P}$, $\mathcal{B}(\mathcal{H}) \leftarrow \mathcal{B}(\mathcal{P})$ and

$$\mathcal{E}^{(u)} \leftarrow \Pi_{\mathcal{P}} \mathcal{E}^{(u)} \Pi_{\mathcal{P}} \tag{44a}$$

$$\mathcal{I} \leftarrow \Pi_{\mathcal{P}}(\mathcal{I}) \tag{44b}$$

$$\rho \leftarrow \Pi_{\mathcal{P}}(\rho). \tag{44c}$$

With this, now $\mathcal{I} > 0$. One can define the completely positive map $\mathcal{N}(x) = \mathcal{I}^{-1/2} x \mathcal{I}^{-1/2}$. Finally, replace

$$\mathcal{E}^{(u)} \leftarrow \mathcal{N} \mathcal{E}^{(u)} \mathcal{N}^{-1} \tag{45a}$$

$$\mathcal{I} \leftarrow \mathcal{N}(\mathcal{I}) = \mathbb{1} \tag{45b}$$

$$\rho \leftarrow \mathcal{N}^{-1}(\rho). \tag{45c}$$

This substitution makes $\sum_{u \in \mathcal{M}} \mathcal{E}^{(u)}(\mathbb{1}) = \mathbb{1}$, while preserving complete positivity and the resulting ρ is the stationary state of the system. This concludes the proof. ◀

Note that several steps in the reduction algorithm could be avoided by imposing further conditions on the properties of the subspaces defining $\mathfrak{P}$, but to explore these relations is beyond the scope of this work.

This constructive algorithm shows that not only appropriate completely positive maps can be obtained from the condition $\mathcal{D} \in \mathfrak{P}$, but also that their structure can be cast into the form of a quantum instrument, where ρ is a fixed point of $\sum_{u \in \mathcal{M}} \mathcal{E}^{(u)}$. The fact that a dimension smaller than that of $\mathcal{B}(\mathcal{H})$ is capable of reproducing the model described by $(\mathcal{B}(\mathcal{H}), \rho, \mathcal{E}, \mathcal{I})$ is ultimately to the non-primitivity of $\mathcal{E}^*$ and the lack of information completeness of the POVM elements $M^{(\mathbf{u})} = \mathcal{E}^{(\mathbf{u})}(\mathcal{I})$. This theorem establishes under which that this explanation is the only possible one, revealing the essential traits that a quasi-realization should exhibit in order to be equivalent to a higher-dimensional quantum model.

7 Discussion

This result represents a generalization of Dharmadhiraki's polyhedral cone condition [9] and establishes the type of positivity that needs to be respected at the level of the regular realization for there to exist a certain lifting in $\mathcal{B}(\mathcal{H})$. The result, highlights a central issue that goes unnoticed in the commutative case. Unlike in the formulation of Dharmadhiraki's cone condition, the truly fundamental object is the set of cones $\mathcal{P}$, from which the cones $\mathcal{C}$ and $\mathcal{C}^*$ can be derived. This shifts the focus from the geometry of the cone of states, and sets it on the nature of the semigroup of transformations corresponding to a given process p.

Of course this is far from a full solution to the problem. Although condition (32) can be verified by a semidefinite program, finding the suitable cone $\mathcal{P}$ for a given process is still a formidable challenge. Our result highlights significant departures from the PRP, so that novel approaches may be possible. In particular, the CPRP turns out to be deeply related to lifting properties for quotient operator systems. Aspects of this theory are deeply connected with several open questions in operator theory [12], such as Connes Embedding Problem and Kirchberg's conjecture. In addition, classical algorithms for learning Hidden Markov Models using matrix factorizations [8] may be extended to semidefinite factorizations [13, 14] thus establishing links between the computational complexity of the CPRP and that of other relevant problems in Quantum Information science. An interesting question, from the operator systems theory point of view, is to identify the abstract operator system in $\mathcal{V}$ for which $\mathcal{P}$ is *the cone* of completely positive maps, and to determine its nuclearity properties.

Just as the positive realization problem, the completely-positive realization problem is highly relevant in systems identification and quantum control. It addresses the problem of finding compact models for systems with quantum memory and a classical readout interface. In particular, modeling stochastic processes which are generated by quantum devices will be the primary application of our results. The positive description of a process not only provides insight into the physical mechanisms underlying a process, but allows to identify *latent* variables, e. g., variables that are not directly observed but allow to draw order and simplicity in otherwise apparently chaotic and highly unpredictable behavior. In this sense, accounting for hidden quantum mechanical mechanisms, and more importantly, quantum memory to an information source, is potentially the difference between obtaining a simple description of a process or a highly complex one.

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